

# Uncertainty relations for the realisation of macroscopic quantum superpositions and EPR paradoxes

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## Abstract

We present a unified approach, based on the use of quantum uncertainty relations, for arriving at criteria for the demonstration of the EPR paradox and macroscopic superpositions. We suggest to view each criterion as a means to demonstrate an EPR-type paradox, where there is an inconsistency between the assumptions of a form of realism, either macroscopic realism (MR) or local realism (LR), and the completeness of quantum mechanics.

## 1 Introduction

Schrödinger [1] raised the question of whether there could be a superposition of macroscopically distinct states. The issue at hand[2] is that where we have a quantum superposition of two states, the system cannot be thought of as being in one state or the other until a measurement is performed that would distinguish the states.

The concept of the quantum superposition is intrinsically associated with the concept of a *fundamental quantum indeterminateness*, that we are limited in the precision to which we can ever predict outcomes of measurements that are performed on the system. This follows because if we have a superposition of two eigenstates  $|x_1\rangle$  and  $|x_2\rangle$  of an observable  $\hat{x}$ , where  $x_2 - x_1$  is large, then by our interpretation, the system is not predetermined to be in either state, so we have an indeterminacy in the outcome  $x$  that is at least of order  $x_2 - x_1$ .

This indeterminacy is of a fundamentally different nature to that of classical theory, where lack of knowledge of an outcome is understood in terms of a statistical theory in which there is a probability for the system to be in a certain state, which will have a certain probability of outcome for  $x$ . Such probabilistic interpretations are generally referred to as *classical mixtures*. In quantum

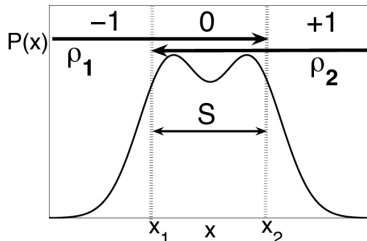


Figure 1: Consider three regions of outcome  $\pm 1, 0$  for measurement  $\hat{x}$ . Density operator  $\rho_1$  encompasses outcomes  $x < x_2$  and  $\rho_2$  encompasses outcomes  $x > x_1$ .

mechanics, the indeterminacy that arises from a quantum superposition is not represented this way.

The concept of a *macroscopic superposition* is therefore linked with that of a *macroscopic quantum indeterminateness*, which manifests as a macroscopic spread in outcomes  $x$  that cannot be explained using statistical mixtures of “smaller” states, that is, states whose predictions give a smaller spread of outcome. The issue of macroscopic quantum indeterminateness is fundamental to quantum mechanics, in that any pure state can be written in terms of eigenstates of any observable, and it is always the case that the uncertainty principle will apply to prevent absolute predetermination of another observable. Put another way, an eigenstate of momentum when written in terms of position eigenstates will be a superposition  $|\psi\rangle = \sum_i c_i |x_i\rangle$  of a macroscopic — in fact infinite — range of position eigenstates  $|x\rangle$ .

In terms of Schrödinger’s concern, we are left to question the real existence of macroscopic quantum indeterminateness, since this would imply a superposition of eigenstates with an inherently macroscopic range of prediction of  $x$ . Following [3], this is still a paradox. We consider two regions of outcome (denoted  $\pm 1$ ) that are macroscopically separated, and denote the region of intermediate outcomes by 0, as shown in Figure 1. The mixture  $\rho = P_1 \rho_1 + P_2 \rho_2$ , where  $\rho_1$  encompasses outcomes  $x < x_2$  and  $\rho_2$  encompasses outcomes  $x > x_1$  ( $P_{1/2}$  are probabilities), imposes a “macroscopic reality”, in the sense that the system can be interpreted to be in possibly one (but never both) of two macroscopically - separated regimes. The macroscopic superpositions defy this assertion.

We present a unified approach for constructing criteria for macroscopic superpositions and EPR entanglement. We first review some experimental signatures[3] for determining the extent of “quantum fuzziness”. These signatures are based on the use of quantum uncertainty relations. Next, we show

how one can easily construct from single-system uncertainty relations new such signatures that apply to bipartite entangled systems. These new signatures result by simply substituting *one* of the variances of the original uncertainty relation with the variance of an inferred observable. Finally we show that the simple further amendment of the uncertainty relations so that all variances are replaced by inferred variances will result in criteria for the EPR paradox[4].

## 2 Macroscopic realism, local realism and the completeness of quantum mechanics

The assumption we seek to test is *macroscopic realism* (MR)[2] — that physical systems can always be described at any given time as being in one or other of two macroscopically distinct states. This can (in principle) coexist with a lack of such realism at the microscopic level.

EPR[4] argued against the completeness of quantum mechanics — the notion that quantum mechanics is a complete theory in the sense that there are no further facts about physical systems which are not captured by a quantum description. In particular, quantum observables obey uncertainty relations and the assumption of completeness implies that the values of those observables are not defined beyond that precision. EPR showed how this assumption of completeness of quantum mechanics clashed with that of *local realism* (LR).

This assumption of the completeness of quantum mechanics does not seem a priori to clash with MR — an argument could be made that the uncertainty principle imposes only a microscopic limitation on the predetermination of observables. We show that this could be a misleading argument, in that quantum mechanics *predicts* the existence of eigenstates of an observable (this observable is said to be *squeezed*) and thus implies infinite spreads in “quantum fuzziness”, for conjugate observables. This prediction we wish to test.

## 3 Criteria for S-scopis superpositions

**Continuous variable case:** We consider a system  $A$  for which an observable  $\hat{x}$  displays a macroscopic range of values. We denote by  $\hat{p}$  the observable conjugate to  $\hat{x}$ , so that (in appropriate units)  $\Delta^2 x \Delta^2 p \geq 1$ .

Leggett and Garg[2] defined *macroscopic realism* (MR) as the assumption: "A macroscopic system with two or more macroscopically distinct states available to it will at all times *be* in one or the other of these states". If we do not want to restrict a priori what states are available to the system, we must assume that all possible superpositions of eigenstates of  $\hat{x}$  are available. If two states each localized around macroscopically distinct values of  $x$  indicate two macroscopically distinct states, then each (pure) quantum state allowed by MR can only have a microscopic (or non-macroscopic) range of outcomes.

In applying MR to situations where more than two states are available, we thus postulate that MR asserts the system to be describable as a statistical

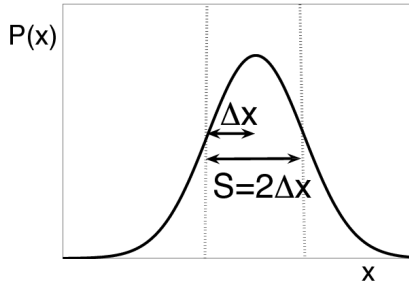


Figure 2: Squeezed states predict a Gaussian distribution for  $x$  with variance  $\Delta x = e^r$ . The measurement of a  $\Delta p$  would imply superpositions of  $|x\rangle$  that have a range (or size)  $S$  where  $S > 2/\Delta p$ . For the squeezed state,  $S > 2\Delta x$ .

mixture of states  $\rho_i^{(S)}$ , each of which predicts a small (non-macroscopic) spread of outcomes  $x$  for  $\hat{x}$ . We now assume that the “states” are *quantum* states, and call this premise *macroscopic quantum realism*. In this case, denoting the spread in the prediction for  $x$  for the state  $\rho_i^{(S)}$  by  $S$ , we can write the density matrix as

$$\rho = \sum_i P_i \rho_i^{(S)} \quad (1)$$

Here  $\sum_i P_i = 1$  and for each  $\rho_i^{(S)}$ ,  $|x_1 - x_2| \leq S$  for all values of outcomes  $x_1$  and  $x_2$  which have zero probability. This assumption leads [3] to constraints on the minimum fuzziness in the conjugate observable  $p$ . Specifically, it follows, since each  $\rho_i^{(S)}$  is itself a quantum state and since the variance predicted by a mixture cannot be less than the average of the variances of its components, that  $\Delta^2 p \geq \frac{4}{S^2}$ .

The experimental observation of squeezing in  $p$  such that  $\Delta p < 2/S$  therefore implies the failure of mixtures of quantum states that can only have a spread in their prediction for  $x$  of  $S$  or less. Thus necessarily the system exists with some probability in a pure superposition state of spread, or size,  $S$  where

$$S > 2/\Delta p \quad (2)$$

The squeezed state[5]  $|\psi\rangle = e^{r(a^2 - a^{\dagger 2})} |0\rangle$  ( $a$  is the boson operator for a field mode at  $A$  and  $|0\rangle$  is the vacuum state) is the simplest model for squeezed variances, defined as  $\Delta p < 1$  (Fig. 2). Here measurements are:  $\hat{x} = (a^\dagger + a)$ ,

$\hat{p} = i(a^\dagger - a)$ . The squeezed state predicts  $\Delta^2 x = \sigma = e^{2r}$ , so that  $x$  has eventually a macroscopic quantum indeterminacy, while  $p$  is squeezed, so that  $\Delta^2 p = 1/\sigma = e^{-2r}$ . Experiments[6, ?, ?] using optical fields have confirmed the existence of squeezed states. Values reported are of order  $\Delta p = 0.4$ , to confirm a quantum superposition of eigenstates  $|x\rangle$  with  $S = 4$ , which is twice that of the coherent state.

**Discrete case:** We present new criteria for the extent of quantum indeterminateness for spin states with discrete outcomes. We use  $\Delta J_X \Delta J_Y \geq |\langle J_Z \rangle|/2$ , where  $J_x, J_Y, J_Z$  are angular momentum observables. Suppose  $\rho$  to be a mixture of superpositions of the eigenstates of  $J_X$  that have an extent  $S$  or less. This leads to the constraint  $\Delta J_Y \geq |\langle J_Z \rangle|/S$ . Thus if we measure a value  $\Delta J_Y$  we can infer existence of superpositions of size  $S$  where

$$S > |\langle J_Z \rangle|/\Delta J_Y \quad (3)$$

The inequality is interesting in that the bound  $|\langle J_Z \rangle|$  itself is not intrinsically restricted in size. This means that it is possible to deduce existence of superpositions of spin eigenstates which have a macroscopic extent in the indeterminateness, even if this extent is small relative to the quantum limit itself.

One example is the observation of squeezing in “spin” observables constructed via the Schwinger formalism. We define  $J_X^A = (a_- a_+^\dagger + a_+^\dagger a_-)/2$ ,  $J_Y^A = (a_- a_+^\dagger - a_+^\dagger a_-)/2i$ ,  $J_Z^A = (a_+^\dagger a_+ - a_-^\dagger a_-)/2$ , where  $a_\pm$  are boson operators for field modes. The physical measurements are of photon number differences, the  $J_X$  and  $J_Y$  measurements being performed by first combining the fields with appropriate phase shifts. Thus, we define  $a_{X\pm} = (a_+ \pm a_-)/\sqrt{2}$  and  $a_{Y\pm} = (a_+ \mp ia_-)/\sqrt{2}$  to get  $J_X = (a_{X+}^\dagger a_{X+} - a_{X-}^\dagger a_{X-})/2$  and  $J_Y = (a_{Y+}^\dagger a_{Y+} - a_{Y-}^\dagger a_{Y-})/2$ . Squeezing of spin variables for the macroscopic regime where outcomes become effectively continuous has been observed, in experiments[7, 8, 10, 9] based on polarisation and atomic-spin squeezing.

## 4 Criteria for S-scopic superpositions in bipartite systems

**Continuous variable case:** We consider two subsystems  $A$  and  $B$ , and define observables  $x, p$  for  $A$ , and  $x^B, p^B$  for  $B$ , where  $\Delta x^B \Delta p^B \geq 1$ . We derive an uncertainty relation that will be useful in deriving signatures for superpositions of entangled systems.

**Theorem 1:** For any quantum state

$$\Delta x \Delta_{inf} p \geq 1 \quad (4)$$

We define the average variance in the inference of  $p$  given a measurement  $\hat{O}^B$  at  $B$  as  $\Delta_{inf}^2 p = \sum_{O^B} P(O^B) \Delta^2(p|O^B)$ :  $\Delta^2(p|O^B)$  is the variance of the conditional distribution  $P(p|O^B)$  and  $P(O^B)$  is the probability of  $O^B$ , the result for observable  $\hat{O}^B$ . In general, where we have a quantum uncertainty relation

of type  $\Delta O_1 \Delta O_2 \geq |\langle [O_1, O_2] \rangle|/2$ , or  $\sum_I \Delta^2 O_I \geq D$ , we can construct another quantum relation that applies to bipartite systems by substituting *one* of the variances,  $\Delta^2 O$  say, for the system  $A$ , with the variance  $\Delta_{inf}^2 O$  of the inferred value for the observable  $\hat{O}$ .

**Proof:** The variance  $\Delta^2 x$  is calculable from the density operator for  $A$  which is  $\rho^A = Tr_B \rho = \sum_{O^B} P(O^B) \rho_{O^B}^B$  where  $\rho_{O^B}^B$  is the reduced state of  $A$  conditional on the result  $O^B$  for the measurement  $\hat{O}^B$  at  $B$ . We thus get  $\Delta^2 x \geq \sum_{O^B} P(O^B) \Delta_{O^B}^2(x|O^B)$ , since the variance of a mixture can't be less than the average of the variances of its components. Here we denote  $\Delta_{O^B}^2(x|O^B)$  as the variance of the conditional  $P(x|O^B)$ . Now using the Cauchy Schwarz inequality

$$\Delta^2 x \Delta_{inf}^2 p \geq \sum_{O^B} P(O^B) \Delta^2(x|O^B) \sum_{O^B} P(O^B) \Delta^2(p|O^B) \quad (5)$$

$$\geq \left[ \sum_{O^B} P(O^B) \Delta(x|O^B) \Delta(p|O^B) \right]^2 \geq 1 \quad (6)$$

Similar reasoning holds for the more general uncertainty relation except that one uses  $\Delta(O_1|O^B) \Delta(O_2|O^B) \geq |\langle C|O^B \rangle|/2$ , where  $C = [O_1, O_2]$  and  $\langle C|O^B \rangle$  denotes the average of  $P(C|O^B)$ , and the fact that in general  $\sum_z P(z) |\langle x|z \rangle| \geq \sum_z P(z) \langle x|z \rangle = \sum_z P(z) \sum_x x P(x|z) = \langle x \rangle$ . The result for the sums of variances can be proved in a similar fashion.

The assumption that  $\rho$  can be expressed as a mixture of *only* S-scopie superpositions of  $|x\rangle$  will imply, following the logic outlined in Section 3, the constraint  $\Delta_{inf} p \geq 2/S$ . The observation of a  $\Delta_{inf} p$  allows us to deduce the existence of a superposition of eigenstates  $|x\rangle$  with a spread  $S$ , where

$$S > 2/\Delta_{inf} p \quad (7)$$

An arbitrary amount of squeezing  $\Delta_{inf} p$  is predicted for the two-mode squeezed state [11, 12]  $|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle_A |n\rangle_B$ , where  $c_n = \tanh^n r / \cosh r$ . Here  $\Delta x = \sigma = \cosh 2r$  while  $\Delta_{inf} p = 1/\cosh 2r$ . The inference variance  $\Delta_{inf} p$  has been measured and recorded in experiments[13] that are designed to test for the EPR paradox. Values as low as  $\Delta_{inf} p \approx 0.7$  have been achieved.

**Discrete case:** We now consider where spin measurements  $J_\theta$  and  $J_\phi^B$  can be performed. Application of Theorem 1 leads to the following inequality satisfied by all such quantum systems:  $\Delta J_X \Delta_{inf} J_Y \geq |\langle J_Z \rangle|/2$ . The observation of a certain inference variance  $\Delta_{inf} J_Y$  will lead to the conclusion of a superpositions of eigenstates of  $J_X$  with spread

$$S > |\langle J_Z \rangle|/\Delta_{inf} J_Y \quad (8)$$

Measurements of  $\Delta_{inf} J_Y$  have been reported by Bowen et al[14].

## 5 Criteria for the EPR paradox

We consider quantum uncertainty relations for system  $A$  of a bipartite system. For example we may have  $\Delta O_1 \Delta O_2 \geq |\langle [O_1, O_2] \rangle|/2$  where  $[O_1, O_2]$  evaluates as another observable which we denote  $C$ . Alternatively, we may have[15]  $\sum_i \Delta^2 O_i \geq D$  where  $D$  is a constant. Because we have a second system  $B$ , we can define the inferred variances  $\Delta_{inf}^2 O_i$ . The following result allows an immediate writing down of criteria to confirm EPR's paradox[4].

**Theorem 2:** Where we have such a quantum uncertainty relation that holds for all quantum states, we can substitute the variances  $\Delta^2 O$  by average inference variances  $\Delta_{inf}^2 O$ , and the mean  $|\langle C \rangle|$  by  $|\langle C \rangle|_{inf}$ , the average inference of the modulus of the mean as defined by  $|\langle C \rangle|_{inf} = \sum_{O^B} P(O^B) |\langle C | O^B \rangle|$ , where  $\langle C | O^B \rangle$  is the mean of the conditional distribution  $P(C | O^B)$ . The resulting inequality is an ‘‘EPR inequality’’ that if violated is a demonstration of the EPR paradox.

**Proof:** We follow the treatment given by EPR[4], and the modifications[12, 16], to conclude existence of an ‘‘element of reality’’  $\mu_{O_i}$  that predetermines the result of measurement for observable  $O_i$ . The probability distribution for the prediction of this element of reality is precisely that of the conditional  $P(O_i | O^B)$  where  $O^B$  is the result of a measurement performed at  $B$ , to infer the value of  $O_i$ . EPR's local realism (LR) implies a joint probability distribution  $P(\lambda)$  for the  $\mu_i$ , or for further underlying parameters. For the product of the inference variances we get, assuming LR

$$\Delta_{inf}^2 O_1 \Delta_{inf}^2 O_2 = \sum_{O_1^B} P(O_1^B) \Delta^2(O_1 | O_1^B) \sum_{O_2^B} P(O_2^B) \Delta^2(O_2 | O_2^B) \quad (9)$$

$$\geq \left[ \sum_{\lambda} P(\lambda) \Delta(O_1 | \lambda) \Delta(O_2 | \lambda) \right]^2 \quad (10)$$

$$\geq \left| \sum_{\lambda} P(\lambda) |\langle C | \lambda \rangle|/2 \right|^2 \geq |\langle C \rangle|_{inf}^2 / 4 \quad (11)$$

and for the sum one obtains

$$\begin{aligned} \Delta_{inf}^2(O_1) + \Delta_{inf}^2(O_2) &= \sum_{O_1^B} P(O_1^B) \Delta^2(O_1 | O_1^B) + \sum_{O_2^B} P(O_2^B) \Delta^2(O_2 | O_2^B) \\ &= \sum_{\lambda} P(\lambda) [\Delta^2(O_1 | \lambda) + \Delta^2(O_2 | \lambda)] \geq D \end{aligned} \quad (12)$$

We have used[12] that if the ‘‘elements of reality’’ can be written as quantum states, then the variances predicted by the elements of reality  $\lambda$  must satisfy the quantum uncertainty relations. This leads to the result (11), once it is realised that increasing the number of variables  $\lambda$  can only decrease the average modulus of the mean. The violation of (11) or (12) thus implies inconsistency of LR with the completeness of quantum mechanics, that the underlying states symbolized by the elements of reality can be quantum states.

The “EPR inequalities”

$$\Delta_{inf}^2 x \Delta_{inf}^2 p \geq 1, \quad \Delta_{inf} J_X \Delta_{inf} J_Y \geq |\langle J_Z \rangle_{inf}|/2 \quad (13)$$

(the latter implies the further EPR inequality[14]  $\Delta_{inf} J_X \Delta_{inf} J_Y \geq |\langle J_Z \rangle|/2$ ) have been derived previously[12] and in some cases used to demonstrate an EPR paradox[13, 14]. One can also use Theorem 2 to derive EPR inequalities from uncertainty relations involving sums of variances, so that for example  $\Delta^2 J_X + \Delta^2 J_Y + \Delta^2 J_Z \geq j/2$  as used by Hoffmann et al[15] leads to the EPR inequality  $\Delta_{inf}^2 J_X + \Delta_{inf}^2 J_Y + \Delta_{inf}^2 J_Z \geq j/2$ .

## 6 Conclusion

The criteria we have derived are based on the assumption that the systems can be described as mixtures of underlying *quantum* states, which therefore satisfy uncertainty relations. This means that the criteria can be viewed in a unified way as conditions for demonstration of general EPR-type paradoxes. In the case of the criteria for macroscopic superpositions, we *assume macroscopic realism* (MR) to infer that the system be described as probabilistic mixture of states with a *microscopic lack of predetermination* only. The assumption that these underlying states be *quantum* states leads to our inequalities. An experimental violation of the inequalities confirms existence of macroscopic superpositions, but does not falsify macroscopic realism itself, since one may propose alternative theories in which the underlying states are *not quantum* states. Hence we have extended the EPR paradox to demonstrate an inconsistency between completeness of quantum mechanics and *macroscopic realism*.

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